Loan Guarantees
Part V - The Revised BSOPM - The Greeks

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In this white paper we will derive the equations for selected Greeks applicable to the uncapped guarantee. We will leave the derivation of the Greeks applicable to the capped guarantee to the ambitious reader. The Greeks that we will derive will be used in Part VI of this series to prove the PDE.

In Part IV of this series we defined the variables d_1, d_2, d_3 and d_4 as follows...

$$d_1 = \left[\ln\left(\frac{D_T}{A_t}\right) - \left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)(T-t)\right] \Bigg/ \sigma\sqrt{T-t}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$
$$d_3 = \left[\ln\left(\frac{D_T - CAP}{\Gamma A_t}\right) - \left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)(T-t)\right] \Bigg/ \sigma\sqrt{T-t}$$
$$d_4 = d_3 - \sigma\sqrt{T-t} \tag{1}$$

In Part IV we defined the functions $f(\alpha,t)$ and $g(\phi,t)$ as follows...

$$f(\alpha,t) = \mathrm{Exp}\left\{-\,\alpha\,(T-t)\right\} \text{ ...and... } g(\phi,t) = \mathrm{Exp}\left\{-\,\phi\,(T-t)\right\} \tag{2}$$

Using Equations (1) and (2) above the equation for the value of an uncapped guarantee from Part IV is...

$$G_t = D_T\,f(\alpha,t)\,CND\Big[d_1\Big] - \Gamma A_t\,g(\phi,t)\,CND\Big[d_2\Big] \tag{3}$$

Delta - Uncapped Guarantee

The first derivative of Equation (3) above with respect to enterprise value is...

$$\frac{\delta G_t}{\delta A_t} = D_T\,f(\alpha,t)\,\frac{\delta CND[d_1]}{\delta A_t} - \Gamma g(\phi,t)\left(CND[d_2] + \frac{\delta CND[d_2]}{\delta A_t}\,A_t\right)$$
$$= D_T\,f(\alpha,t)\,\frac{\delta CND[d_1]}{\delta A_t} - \Gamma g(\phi,t)\,CND[d_2] - \Gamma A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta A_t} \tag{4}$$

Note the following equality from Equation (27) below...

$$A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta A_t} = D_T\,f(\alpha,t)\,\frac{\delta CND[d_1]}{\delta A_t} \tag{5}$$

Using Equation (5) above we can rewrite Equation (4) above as...

$$\frac{\delta G_t}{\delta A_t} = A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta A_t} - \Gamma g(\phi,t)\,CND[d_2] - \Gamma A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta A_t}$$
$$= (1-\Gamma)\,A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta A_t} - \Gamma g(\phi,t)\,CND[d_2] \tag{6}$$

Using the chain rule we can rewrite Equation (6) above as...

$$\frac{\delta G_t}{\delta A_t} = (1-\Gamma)\,A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\delta d_2}{\delta A_t} - \Gamma g(\phi,t)\,CND[d_2] \tag{7}$$

Using Appendix Equations (40) and (41) below we can rewrite Equation (7) above as...

$$\begin{aligned}\frac{\delta G_t}{\delta A_t} &= (1-\Gamma)\,A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\times\frac{-1}{\sigma\sqrt{T-t}}\,A_t^{-1}-\Gamma g(\phi,t)\,CND[d_2]\\ &= (\Gamma-1)\,g(\phi,t)\,\frac{1}{\sigma\sqrt{T-t}}\,\frac{\delta CND[d_2]}{\delta d_2}-\Gamma g(\phi,t)\,CND[d_2]\end{aligned} \tag{8}$$

Note that the functions in Equation (8) above are referenced below...

$$g(\phi,t)\ :\ \text{See Equation (39)}\ ;\ \frac{\delta CND[d_2]}{\delta d_2}\ :\ \text{See Equation (44)}\ ;\ CND[d_2]\ :\ \text{See Equation (44)} \tag{9}$$

Gamma - Uncapped Guarantee

The second derivative of Equation (3) above with respect to enterprise value is...

$$\frac{\delta^2 G_t}{\delta A_t^2}=\frac{\delta}{\delta A_t}\left[(\Gamma-1)\,g(\phi,t)\,\frac{1}{\sigma\sqrt{T-t}}\,\frac{\delta CND[d_2]}{\delta d_2}-\Gamma g(\phi,t)\,CND[d_2]\right] \tag{10}$$

Using the chain rule we can rewrite Equation (10) above as...

$$\begin{aligned}\frac{\delta^2 G_t}{\delta A_t^2} &= \frac{\delta}{\delta d_2}\left[(\Gamma-1)\,g(\phi,t)\,\frac{1}{\sigma\sqrt{T-t}}\,\frac{\delta CND[d_2]}{\delta d_2}\right]\frac{\delta d_2}{\delta A_t}-\Gamma g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\delta d_2}{\delta A_t}\\ &= (\Gamma-1)\,g(\phi,t)\,\frac{1}{\sigma\sqrt{T-t}}\,\frac{\delta^2 CND[d_2]}{\delta d_2^2}\,\frac{\delta d_2}{\delta A_t}-\Gamma g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\delta d_2}{\delta A_t}\end{aligned} \tag{11}$$

Using Appendix Equations (40) and (41) below we can rewrite Equation (11) above as...

$$\begin{aligned}\frac{\delta^2 G_t}{\delta A_t^2} &= (\Gamma-1)\,g(\phi,t)\,\frac{1}{\sigma\sqrt{T-t}}\,\frac{\delta^2 CND[d_2]}{\delta d_2^2}\,\frac{-1}{\sigma\sqrt{T-t}}\,A_t^{-1}-\Gamma g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\delta d_2}{\delta A_t}\\ &= (1-\Gamma)\,g(\phi,t)\,\frac{1}{\sigma\sqrt{T-t}}\,\frac{\delta^2 CND[d_2]}{\delta d_2^2}\,\frac{1}{\sigma\sqrt{T-t}}\,A_t^{-1}\,\frac{A_t}{A_t}-\Gamma g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\delta d_2}{\delta A_t}\\ &= (1-\Gamma)\,g(\phi,t)\,\frac{A_t}{\sigma\sqrt{T-t}}\,\frac{\delta^2 CND[d_2]}{\delta d_2^2}\,\frac{1}{\sigma\sqrt{T-t}}\,A_t^{-2}-\Gamma g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\delta d_2}{\delta A_t}\end{aligned} \tag{12}$$

Using Appendix Equations (40) and (41) below note the following...

$$\frac{\delta^2 d_2}{\delta A_t^2}=\frac{1}{\sigma\sqrt{T-t}}\,A_t^{-2} \tag{13}$$

Using Equation (13) above we can rewrite Equation (12) above as...

$$\frac{\delta^2 G_t}{\delta A_t^2}=(1-\Gamma)\,g(\phi,t)\,\frac{A_t}{\sigma\sqrt{T-t}}\,\frac{\delta^2 CND[d_2]}{\delta d_2^2}\,\frac{\delta^2 d_2}{\delta A_t^2}-\Gamma g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\delta d_2}{\delta A_t} \tag{14}$$

Note that the functions in Equation (14) above are referenced below...

$$g(\phi,t)\ :\ \text{See Equation (39)}\ ;\ \frac{\delta CND[d_2]}{\delta d_2}\ :\ \text{See Equation (44)}\ ;\ \frac{\delta^2 CND[d_2]}{\delta d_2^2}\ :\ \text{See Equation (46)} \tag{15}$$

$$\frac{\delta d_2}{\delta A_t}\ :\ \text{See Equation (41)}\ ;\ \frac{\delta^2 d_2}{\delta A_t^2}\ :\ \text{See Equation (41)} \tag{16}$$

Theta - Uncapped Guarantee

The derivative of Equation (3) above with respect to time is...

$$\frac{\delta G_t}{\delta t}=D_T\left(\frac{\delta f(\alpha,t)}{\delta t}\,CND\Big[d_1\Big]+\frac{\delta CND[d_1]}{\delta t}\,f(\alpha,t)\right)-\Gamma A_t\left(\frac{\delta g(\phi,t)}{\delta t}\,CND\Big[d_2\Big]+\frac{\delta CND[d_2]}{\delta t}\,g(\phi,t)\right) \tag{17}$$

Using Appendix Equations (38) and (39) below we can rewrite Equation (17) above as...

$$\frac{\delta G_t}{\delta t} = \alpha\, D_T\, f(\alpha,t)\, CND\Big[d_1\Big] + D_T\, f(\alpha,t)\, \frac{\delta CND[d_1]}{\delta d_t} - \Gamma\phi\, A_t\, g(\phi,t)\, CND\Big[d_2\Big] - \Gamma A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta d_t} \tag{18}$$

Using Equation (1) above note the following...

$$\text{if... } d_2 = d_1 - \sigma\sqrt{T-t} \text{ ...then... } \frac{\delta d_2}{\delta t} = \frac{\delta d_1}{\delta t} + \frac{\sigma}{2\sqrt{T-t}} \tag{19}$$

Using Equation (19) above and the chain rule we can rewrite Equation (18) above as...

$$\begin{aligned}\frac{\delta G_t}{\delta t} &= \alpha\, D_T\, f(\alpha,t)\, CND\Big[d_1\Big] + D_T\, f(\alpha,t)\, \frac{\delta CND[d_1]}{\delta d_1}\, \frac{\delta d_1}{\delta t} - \Gamma A_t\, \phi\, g(\phi,t)\, CND\Big[d_2\Big] \\ &- \Gamma A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta d_2} \left(\frac{\delta d_1}{\delta t} + \frac{\sigma}{2\sqrt{T-t}}\right) \\ &= \alpha\, D_T\, f(\alpha,t)\, CND\Big[d_1\Big] + D_T\, f(\alpha,t)\, \frac{\delta CND[d_1]}{\delta d_1}\, \frac{\delta d_1}{\delta t} - \Gamma\phi\, A_t\, g(\phi,t)\, CND\Big[d_2\Big] \\ &- \Gamma A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta d_2}\, \frac{\delta d_1}{\delta t} - \Gamma A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta d_2}\, \frac{\sigma}{2\sqrt{T-t}}\Bigg)\end{aligned} \tag{20}$$

Note that per Equation (30) below...

$$A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta d_2} = D_T\, f(\alpha,t)\, \frac{\delta CND[d_1]}{\delta d_1} \tag{21}$$

Using Equation (21) above we can rewrite Equation (20) above as...

$$\begin{aligned}\frac{\delta G_t}{\delta t} &= \alpha\, D_T\, f(\alpha,t)\, CND\Big[d_1\Big] + (1-\Gamma)\, D_T\, f(\alpha,t)\, \frac{\delta CND[d_1]}{\delta d_1}\, \frac{\delta d_1}{\delta t} - \Gamma\phi\, A_t\, g(\phi,t)\, CND\Big[d_2\Big] \\ &- \Gamma A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta d_2}\, \frac{\sigma}{2\sqrt{T-t}}\end{aligned} \tag{22}$$

After combining terms we can rewrite Equation (22) above as...

$$\frac{\delta G_t}{\delta t} = D_T\, f(\alpha,t) \left(\alpha\, CND\Big[d_1\Big] + (1-\Gamma)\, \frac{\delta CND[d_1]}{\delta d_1}\, \frac{\delta d_1}{\delta t}\right) - \Gamma A_t\, g(\phi,t) \left(\phi\, CND\Big[d_2\Big] + \frac{\delta CND[d_2]}{\delta d_2}\, \frac{\sigma}{2\sqrt{T-t}}\right) \tag{23}$$

Note that the functions in Equation (23) above are referenced below...

$$f(\alpha,t) \text{ : See Equation (38) ; } g(\phi,t) \text{ : See Equation (39) ; } CND\Big[d_1\Big] \text{ : See Equation (44)} \tag{24}$$

$$CND\Big[d_2\Big] \text{ : See Equation (44) ; } \frac{\delta CND[d_1]}{\delta d_1} \text{ : See Equation (44) ; } \frac{\delta CND[d_2]}{\delta d_2} \text{ : See Equation (44)} \tag{25}$$

$$\frac{\delta d_1}{\delta t} \text{ : See Equation (40)} \tag{26}$$

Simplifying Equation Proof

We will prove the following equation...

$$A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta A_t} = D_T\, f(\alpha,t)\, \frac{\delta CND[d_1]}{\delta A_t} \tag{27}$$

Using the chain rule we can rewrite Equation (27) above as...

$$A_t\, g(\phi,t)\, \frac{\delta CND[d_2]}{\delta d_2}\, \frac{\delta d_2}{\delta A_t} = D_T\, f(\alpha,t)\, \frac{\delta CND[d_1]}{\delta d_1}\, \frac{\delta d_1}{\delta A_t} \tag{28}$$

Using Appendix Equation (41) below note the following...

$$\frac{\delta d_1}{\delta A_t}=\frac{\delta d_2}{\delta A_t} \tag{29}$$

Using Equation (29) above we can rewrite Equation (28) above as...

$$A_t\,g(\phi,t)\,\frac{\delta CND[d_2]}{\delta d_2}=D_T\,f(\alpha,t)\,\frac{\delta CND[d_1]}{\delta d_1} \tag{30}$$

Using Appendix Equation (44) below we can rewrite Equation (30) above as...

$$\begin{aligned}A_t\,g(\phi,t)\,\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_2^2\right\}&=D_T\,f(\alpha,t)\,\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_1^2\right\}\\ A_t\,g(\phi,t)\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_2^2\right\}&=D_T\,f(\alpha,t)\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_1^2\right\}\end{aligned} \tag{31}$$

Using Equation (1) above note the following...

$$d_2=d_1-\sigma\sqrt{T-t}\ \text{ ...such that... }\ d_2^2=d_1^2-2\,d_1\sigma\sqrt{T-t}+\sigma^2(T-t) \tag{32}$$

Using Equation (31) above we can rewrite Equation (30) above as...

$$\begin{aligned}A_t\,g(\phi,t)\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_1^2\right\}\mathrm{Exp}\left\{d_1\sigma\sqrt{T-t}-\frac{1}{2}\,\sigma^2(T-t)\right\}&=D_T\,f(\alpha,t)\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_1^2\right\}\\ A_t\,g(\phi,t)\,\mathrm{Exp}\left\{d_1\sigma\sqrt{T-t}-\frac{1}{2}\,\sigma^2(T-t)\right\}&=D_T\,f(\alpha,t)\end{aligned} \tag{33}$$

Using Equation (2) above we can rewrite Equation (32) above as...

$$A_t\,\mathrm{Exp}\left\{-\phi\,(T-t)\right\}\mathrm{Exp}\left\{d_1\sigma\sqrt{T-t}-\frac{1}{2}\,\sigma^2(T-t)\right\}=D_T\,\mathrm{Exp}\left\{-\alpha\,(T-t)\right\} \tag{34}$$

If we take the log of both sides of Equation (33) above then we get the following equation...

$$\ln\left(A_t\right)-\phi\,(T-t)+d_1\sigma\sqrt{T-t}-\frac{1}{2}\,\sigma^2(T-t)=\ln\left(D_t\right)-\alpha\,(T-t) \tag{35}$$

Using Equation (1) above note the following...

$$d_1\sigma\sqrt{T-t}=\ln\left(D_T\right)-\ln\left(A_t\right)-\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)(T-t) \tag{36}$$

Using Equation (35) above we can rewrite Equation (35) above as...

$$\begin{aligned}\ln\left(A_t\right)-\phi\,(T-t)+\ln\left(D_T\right)-\ln\left(A_t\right)-\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)(T-t)-\frac{1}{2}\,\sigma^2(T-t)&=\ln\left(D_t\right)-\alpha\,(T-t)\\ \ln\left(D_t\right)-\alpha\,(T-t)&=\ln\left(D_t\right)-\alpha\,(T-t)\\ 0&=0\end{aligned} \tag{37}$$

Using the result of Equation (37) above we have proved Equation (27) above.

Appendix

A. Using Equation (2) above the derivatives of the function $f(\alpha,t)$ with respect to time and enterprise value are...

$$\frac{\delta f(\alpha,t)}{\delta t}=\alpha\,\mathrm{Exp}\left\{-\alpha\,(T-t)\right\}\ \text{ ...and... }\ \frac{\delta f(\alpha,t)}{\delta A_t}=0 \tag{38}$$

B. Using Equation (2) above the derivatives of the function $g(\phi, t)$ with respect to time and enterprise value are...

$$\frac{\delta g(\phi,t)}{\delta t} = \phi \,\mathrm{Exp}\left\{ -\phi\,(T-t)\right\} \text{ ...and... } \frac{\delta g(\phi,t)}{\delta A_t} = 0 \tag{39}$$

C. Using Equation (1) above the derivatives of the variable d_1 with respect to time and enterprise value are...

$$\frac{\delta d_1}{\delta t} = \left\{ \left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)\sigma\sqrt{T-t} + \left[\ln\left(\frac{D_T}{A_t}\right) - \left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)(T-t)\right] \frac{\sigma}{2\sqrt{T-t}} \right\} \div \sigma^2 (T-t) \text{ ...and...}$$

$$\frac{\delta d_1}{\delta A_t} = -\frac{1}{\sigma\sqrt{T-t}}\,A_t^{-1} \text{ ...and... } \frac{\delta^2 d_1}{\delta A_t^2} = \frac{1}{\sigma\sqrt{T-t}}\,A_t^{-2} \tag{40}$$

D. Using Equations (1) and (40) above the derivatives of the variable d_2 with respect to time and enterprise value are...

$$\frac{\delta d_2}{\delta t} = \frac{\delta d_1}{\delta t} + \frac{\sigma}{2\sqrt{T-t}} \text{ ...and... } \frac{\delta d_2}{\delta A_t} = \frac{\delta d_1}{\delta A_t} \text{ ...and... } \frac{\delta^2 d_2}{\delta A_t^2} = \frac{\delta^2 d_1}{\delta A_t^2} \tag{41}$$

E. Using Equation (1) above the derivatives of the variable d_3 with respect to time and enterprise value are...

$$\frac{\delta d_3}{\delta t} = \left\{ \left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)\sigma\sqrt{T-t} + \left[\ln\left(\frac{D_T - CAP}{\Gamma A_t}\right) - \left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)(T-t)\right] \frac{\sigma}{2\sqrt{T-t}} \right\} \div \sigma^2 (T-t)$$

$$\text{...and... } \frac{\delta d_3}{\delta A_t} = \frac{\delta d_1}{\delta A_t} \tag{42}$$

F. Using Equations (1) and (42) above the derivatives of the variable d_4 with respect to time and enterprise value are...

$$\frac{\delta d_4}{\delta t} = \frac{\delta d_3}{\delta t} + \frac{\sigma}{2\sqrt{T-t}} \text{ ...and... } \frac{\delta d_4}{\delta A_t} = \frac{\delta d_3}{\delta A_t} \tag{43}$$

G. The derivative of the cumulative normal distribution function of d_x with respect to d_x is...

$$CND\left[d_x\right] = \int_{-\infty}^{d_x} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\delta Z \text{ ...where... } \frac{\delta CND[d_x]}{\delta d_x} = \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_x^2\right\} \tag{44}$$

H. The derivatives of Equation (44) above respect to time and enterprise value are...

$$\frac{\delta CND[d_x]}{\delta t} = \frac{\delta CND[d_x]}{\delta d_1}\,\frac{\delta d_x}{\delta t} \text{ ...and... } \frac{\delta CND[d_x]}{\delta A_t} = \frac{\delta CND[d_x]}{\delta d_x}\,\frac{\delta d_x}{\delta A_t} \tag{45}$$

I. The second derivative of Equation (44) above respect to d_x is...

$$\frac{\delta^2 CND[d_x]}{\delta d_x^2} = \frac{\delta}{\delta x}\left[\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_x^2\right\}\right] = -d_x\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,d_x^2\right\} = -d_x\,\frac{\delta CND[d_x]}{\delta d_x} \tag{46}$$

J. Using Equation (46) above the second derivative of Equation (44) above with respect to enterprise value is...

$$\frac{\delta^2 CND[d_x]}{\delta A_t^2} = \frac{\delta^2 CND[d_x]}{\delta d_x^2}\,\frac{\delta d_x}{\delta A_t} + \frac{\delta CND[d_x]}{\delta d_x}\,\frac{\delta^2 d_x}{\delta A_t^2} \tag{47}$$